

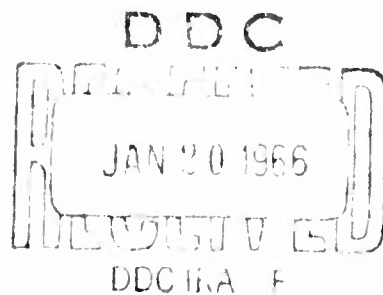
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LINEAR FILTER OPTIMIZATION WITH GAME THEORY CONSIDERATIONS

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M. C. YOVITS
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LINEAR FILTER OPTIMIZATION
WITH GAME THEORY CONSIDERATIONS*†

by

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Summary

The optimum reproduction of a signal in the presence of noise by means of linear filters is considered when the signal is unknown. The problem is likened to a game. In such a game the signal spectrum can be considered to be the strategy of one of the participants, while the strategy of the other participant is specified by the transfer function of the filter. The payoff is taken to be the mean square difference between filter output and signal. The signal producer by his choice of spectrum attempts to maximize the difference while the filter designer attempts to minimize it.

In order to obtain game theory solutions the optimum transfer function for any fixed signal spectrum and also the optimum signal spectrum for any allowable transfer function are found. The game theory solution is then the intersection of these two functional equations.

In order to obtain the optimum transfer function for any fixed signal spectrum in a convenient form a new variational procedure has been developed which yields an integral equation for the amplitude of the transfer function. The form of the relationship is dependent only on the noise spectrum. For simple noise spectra the results are both simple and convenient for game theory solutions.

The new variational procedure enables one easily to find the optimum transfer function when various of the mean square time derivatives of the filter output are fixed. This is shown to be formally equivalent to finding the optimum transfer function for polynomial noise spectra. Detailed calculations of the game theory solution for the case where the mean square second time derivatives of both the filter output and the signal are fixed, have been performed. The mean square difference has been computed as a function of the ratio of the output to signal second derivatives. This is presented graphically.

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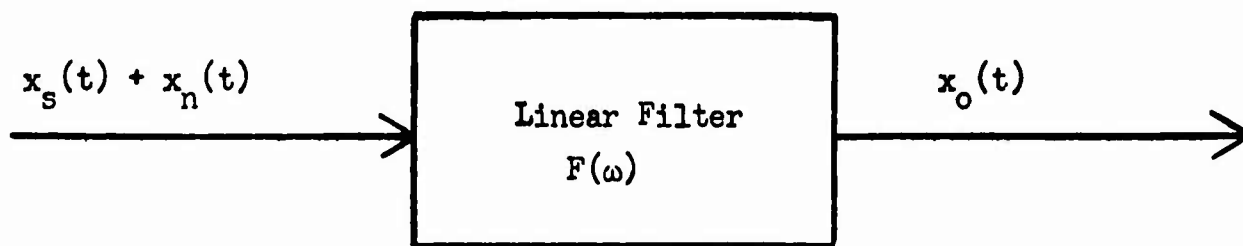
† Presented at the I.R.E. National Convention, New York City, March 21-24, 1955. Now contained in the 1955 National I.R.E. Convention Record, Part 4, Computers, Information Theory, Automatic Control, page 193.

‡ Now on leave of absence with the Office of Naval Research, Washington, D. C.

Introduction

The problem which is considered in this paper is that of following a signal in the presence of noise as closely as possible with a linear filter. The filter input, which is a function of time, is in general made up of both signal, $x_s(t)$, and noise, $x_n(t)$. It is desired to choose a linear filter, the output of which, $x_o(t)$, matches the input signal. Due to the presence of the noise in the input it is clearly impossible for $x_o(t)$ to be identical to $x_s(t)$. The object, then, is to choose that linear filter which matches the two as closely as possible. A convenient criterion for an optimum filter is the one which minimizes the mean square difference between the filter output and the input signal.

Schematically the system is depicted in Fig. 1.



The Basic System

Fig. 1

The filter is considered to be a box whose characteristics are to be chosen in order to minimize the mean square difference between $x_o(t)$ and $x_s(t)$.

The input signal and noise are taken to be stationary random functions; that is, the probability distributions which describe the time functions are invariant with respect to a change of the time zero. Further, the signal and noise are taken to be uncorrelated.

For such a system, the mean square difference between the filter output and the input signal, the error, is given by

$$\sigma^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[S(\omega) |1 - F(\omega)|^2 + N(\omega) |F(\omega)|^2 \right] d\omega, \quad (1)$$

where $S(\omega)$ is the spectral density of the signal and $N(\omega)$ is the spectral density of the noise. It is seen that the miss is made up of two parts. The first part, $\int_{-\infty}^{\infty} S(\omega) |1 - F(\omega)|^2 d\omega$, is the miss which is obtained because the transfer function differs from unity and consequently does not follow the signal exactly. The second part, $\int_{-\infty}^{\infty} N(\omega) |F(\omega)|^2 d\omega$, is the miss which is obtained because the system does follow the noise.

Eq. (1) is a standard equation, the derivation of which is immediate when the signal, noise, and output are transformed to their Fourier components. The mean square error is then found by squaring the absolute difference between the output and the signal and integrating over all frequencies. The (2π) is a normalization factor which appears because of the way in which the power spectrum is defined.

It is desired to minimize the error of Eq. (1) by the proper choice of a realizable transfer function, the so-called optimum transfer function. In previous work in this field as developed by Wiener et al,¹ it is necessary in order to find the optimum transfer function that both the signal and noise spectral densities be known.

This paper however is concerned with the problem of finding the optimum filter over a class of possible signal spectral densities. Specifically, the situation is considered where the producer of the signal by his choice of spectrum attempts to maximize the error. Whatever is producing the signal prefers not to be followed by the filter. The signal spectral density is thus no longer to be regarded as a fixed function, given for the problem, but rather as the strategy of one of the participants of a game. The strategy of the other participant, the designer of the filter, is specified by the transfer function of the filter. The payoff of the game is taken to be the mean square difference between output and input signal. The problem of the filter designer thus becomes one of designing the best filter in view of what, from his viewpoint, is the worst choice of the signal spectral density by the signal producer. From the viewpoint of the signal producer, a signal is generated which has statistical properties that will result in the greatest possible miss despite the best possible efforts of the filter designer. If these two considerations are mutually compatible, then there is said to be a game theory solution and the functions which satisfy these requirements are that solution.

In order to obtain game theory solutions the optimum transfer function for any fixed signal spectrum and also the optimum signal spectrum for any allowable transfer function are found. The game theory solution is then the intersection of these two functional equations.

To obtain the optimum transfer function for any fixed spectrum in a form convenient for game theory analysis a new variational procedure has been developed which yields an integral equation for the amplitude of the transfer function. For simple types of noise spectra the resulting integral equation is easily solved. The functional form of the solution is dependent only on the functional form of the noise spectrum. The resulting solutions have a form which make them especially susceptible to game theory analysis.

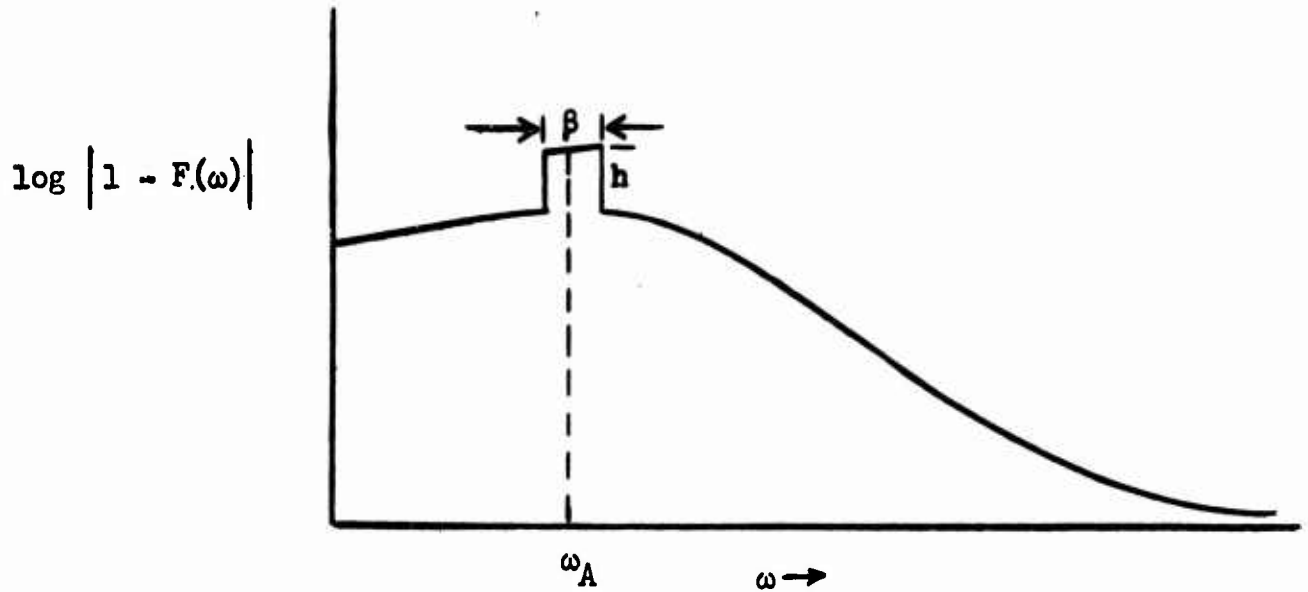
The method utilized is basically a simple one involving only elementary mathematics. In the hope that it will provide additional insight into the general optimum filter problem it is sketched here. It should be noted that this method provides a rigorous and straightforward way of obtaining the optimum transfer function for the cases where the noise spectral density increases at large frequencies with increasing frequency. Such a situation is of more than academic importance, as will be shown later.

The Optimum Transfer Function

It is desired to minimize the mean square error of Eq. (1) by the proper choice of realizable transfer function. If σ^2 is to be a minimum it must be stationary with respect to any small realizable variation in the transfer function, $F(\omega)$. For game theory analysis it is found to be necessary to apply the variation to $1 - F(\omega)$ rather than to $F(\omega)$ itself. For convenience $1 - F(\omega)$ is called $G(\omega)$. The phase which accompanies $G(\omega)$ is said to be $\phi(\omega)$. The vectors $F(\omega)$ and $G(\omega)$ can be considered to be two sides of a triangle, the third side of which is unity. They are thus related to each other by the law of cosines. Consequently Eq. (1) becomes

$$\sigma^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ [S(\omega) + N(\omega)] \cdot |G(\omega)|^2 + N(\omega) - 2N(\omega) |G(\omega)| \cos \phi(\omega) \right\} d\omega. \quad (1a)$$

A general variation in the log of $|G(\omega)|$ which can conveniently be used is indicated in Fig. 2. The variation of height h and width β is applied at an arbitrary frequency ω_A .



General Variation in Log $|1 - F(\omega)|$

Fig. 2

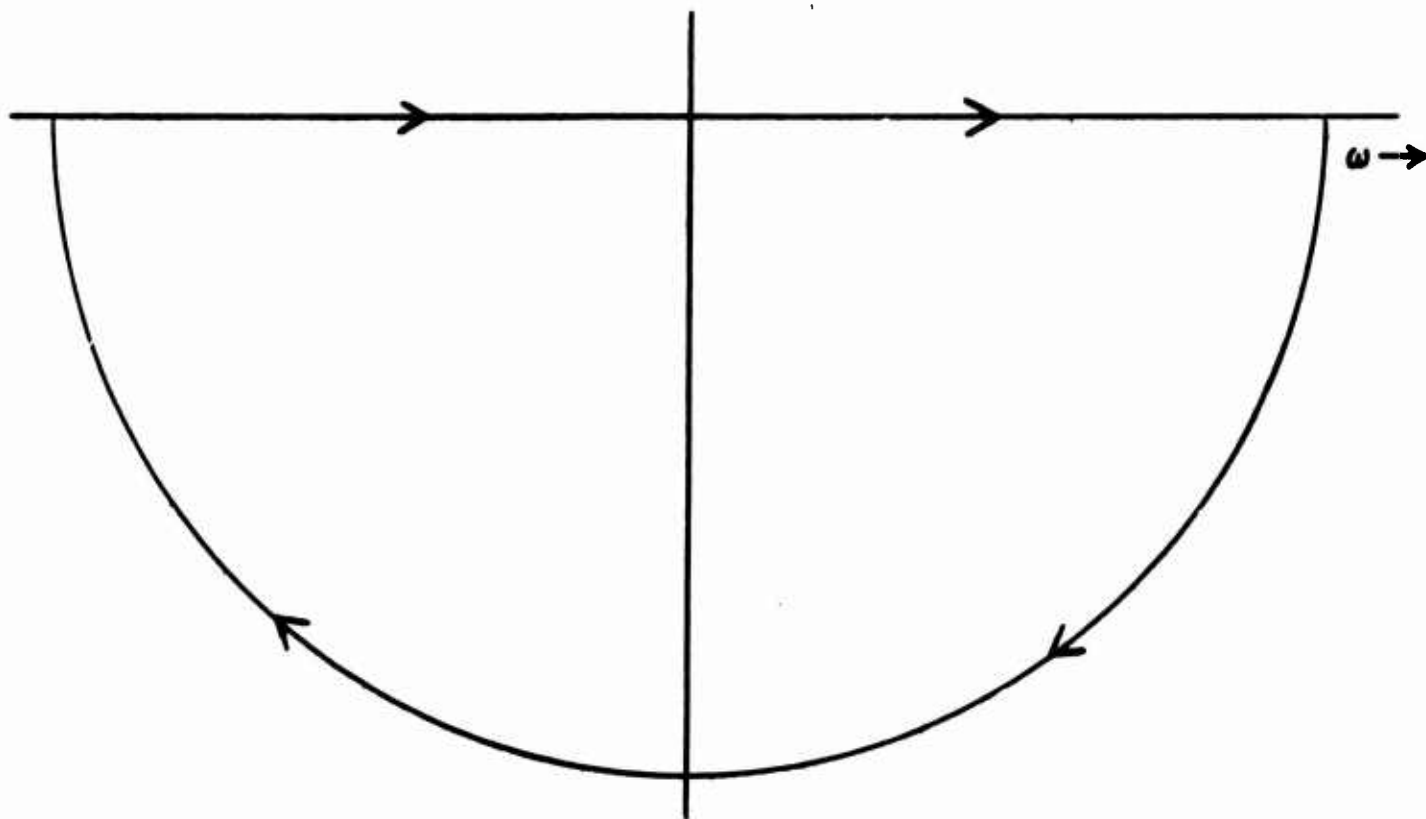
Since the variation must be a realizable one, the change in phase associated with it is given from the usual criterion for realizability.² Algebraically,

$$\delta\phi = -\frac{2h\beta}{\pi} \frac{\omega}{\omega^2 - \omega_A^2} \quad (2)$$

If one sets the first variation of σ^2 equal to zero, the following integral equation results:

$$\begin{aligned} \left\{ |G|^2 (S + N) - N |G| \cos \phi \right\}_{\omega = \omega_A} &= \frac{2}{\pi} \int_0^{\infty} \frac{\omega N |G| \sin \phi}{\omega^2 - \omega_A^2} d\omega \\ &= \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\omega N G}{\omega^2 - \omega_A^2} d\omega, \end{aligned} \quad (3)$$

where it is now understood that G , S , and N are all functions of ω . The right side of Eq. (3) for an analytically given noise spectrum is readily evaluated by integration in the complex ω -plane around the contour shown in Fig. 3. The radius of the semi-circle is very large. Since $G(\omega)$ has no singularities in the lower half plane, the integral can be evaluated in terms of the poles of $N(\omega)$ in this region.



The Basic Contour

Fig. 3

Simplification results when it is noted that the contribution from the poles $\pm \omega_A$ is identically $\left\{ -N |G| \cos \varphi \right\}_{\omega_A}$. Thus an expression results

which gives $|G|^2 (S + N)$ in terms of the values of G at the poles of N .

For the case where $N = N_0$, a constant (white noise), it is seen that the solution for the optimum transfer function is

$$N(\omega) = N_0 ; \quad |G(\omega)|^2 = |1 - F(\omega)|^2 = \frac{N(\omega)}{N(\omega) + S(\omega)} . \quad (4)$$

For Markoffian noise, $N(\omega) = \frac{N_0 a^2}{\omega^2 + a^2}$, the optimum transfer function is readily given as

$$N(\omega) = \frac{N_0 a^2}{\omega^2 + a^2} ; \quad |G(\omega)|^2 = |1 - F(\omega)|^2 = K_a \frac{N(\omega)}{N(\omega) + S(\omega)} ; \quad (5)$$

where

$$K_a = \exp \left[\frac{2}{\pi N_0 a} \int_0^\infty N(\omega) \log \frac{N(\omega)}{N(\omega) + S(\omega)} d\omega \right] . \quad (5a)$$

More complicated types of noise spectra give similar albeit, more complicated, results. Should it be desired to obtain the actual transfer function, $F(\omega)$, this can be done from the law of cosines where the phase is obtained uniquely from the realizability criterion. It can be shown that for simple noise spectra like those above, $G(\omega)$ must be minimum phase.

The above results are of course in agreement, although in different forms, with the results as given by Wiener.¹

One can integrate each term of Eq. (3) with respect to ω_A from $-\infty$ to $+\infty$. If $N(\omega)$ goes to zero at infinity, the order of integration of the double integral on the right hand side of the equation can be interchanged and is then seen to be identically zero. The relation is obtained then for the optimum $|G|^2$, where ω_A is now called φ

$$N(\omega) \rightarrow 0 ; \quad \int_0^\infty |G|_{\text{opt}}^2 (S + N) d\omega = \int_0^\infty N |G|_{\text{opt}} \cos \varphi d\omega . \quad (6)$$

$\omega \rightarrow \infty$

This leads to the relationship for the optimum error

$$N(\omega) \rightarrow 0 ; \quad \sigma_{\text{opt}}^2 = \frac{1}{\pi} \int_0^\infty \left[N - |G|_{\text{opt}}^2 (S + N) \right] d\omega . \quad (6a)$$

$\omega \rightarrow \infty$

As will be shown later, cases where the noise spectral density increases at infinite frequency are of importance. For such cases, the integral of Eq. (3) does not converge. It is necessary to use a less general variation than that shown in Fig. 2; i.e., a dipole, quadrupole, or higher order variation depending on the behavior of the noise spectrum at infinity. A slightly different variational equation for the optimum transfer function results.

For example, when the noise spectrum is a polynomial of order ω^4 then the variational equation to be solved is

$$N(\omega) = N_0 + N_2 \omega^2 + N_4 \omega^4 ;$$

$$\left\{ |G|^2 (S + N) - N |G| \cos \phi \right\}_{\omega=0}^{\omega_A} = \frac{2}{\pi} \omega_A^2 \int_0^{\omega} \frac{N |G| \sin \phi}{\omega(\omega^2 - \omega_A^2)} d\omega . \quad (7)$$

This is readily solved to give for the optimum transfer function

$$N(\omega) = N_0 + N_2 \omega^2 + N_4 \omega^4 ; \quad |G(\omega)|^2 = |1 - F(\omega)|^2 = \frac{N(\omega) + K}{N(\omega) + S(\omega)} , \quad (8)$$

where K is given by

$$\int_0^{\omega} \log \frac{N(\omega) + K}{N(\omega) + S(\omega)} d\omega = 0 . \quad (8a)$$

Similar results are obtained for polynomial noise spectra of other orders.

It is important to note that the functional form of the relationship for the optimum $|G(\omega)|^2$ is dependent only on the functional form of the noise spectrum.

For noise which does not decrease at infinity, Eqs. (6) and (6a) are no longer valid. It can be shown that Eq. (6a) then becomes

$N(\omega)$ is a polynomial;

$$\sigma_{\text{opt}}^2 = -\frac{1}{\pi} \int_0^{\infty} \left[N - |G|_{\text{opt}}^2 (S + N) + N \log |G|_{\text{opt}}^2 \right] d\omega . \quad (8b)$$

Note for the case of white noise,

$$N = N_0; \quad \sigma_{\text{opt}}^2 = -\frac{1}{\pi} \int_0^{\infty} N_0 \log \frac{N_0}{N_0 + S(\omega)} d\omega . \quad (8c)$$

Optimum Signal Spectrum and Game Theory Solutions

That transfer function which minimizes the mean square error for any given stationary input signal has now been found. This transfer function can be considered to be the strategy of one of the participants of a game. The strategy of the other participant can be considered to be the input signal spectrum. This participant attempts to maximize the error for any given transfer function.

Clearly the input spectral density which maximizes the mean square error is dependent on the restrictions imposed on this spectrum. If there are no restrictions placed on the input then evidently the error is a maximum when the spectrum is everywhere infinite.

If the mean square value of the n^{th} time derivative of the signal is fixed to be some constant, M , then it can be shown that $S(\omega)$ must have the form

$$S(\omega) = \frac{2\pi M^2}{\omega^{2n}} g(\omega) , \quad (9)$$

where $g(\omega)$ is the signal choice function defined so that

$$\int_{-\infty}^{\infty} g(\omega) d\omega = 1 . \quad (9a)$$

The signal producer desires to choose $g(\omega)$ in such a way as to maximize the mean square error for any given transfer function. It is clear upon examination of Eq. (1) that this is accomplished if the signal spectrum has a value only at the maximum values of $|G(\omega)|^2/\omega^{2n}$ and is elsewhere zero. The intersection between this functional relationship and the applicable relationship for the optimum $|G(\omega)|^2$ for any input signal spectrum is said to be the game theory solution.

For the simpler noise spectral densities which have already been considered it is easily shown that the solution exists and is unique. It is easily shown that the only conditions under which a solution exists is for $|G(\omega)|^2/\omega^{2n}$ to have a broad maximum which extends from $\omega = 0$ to some ω_0 , the cutoff frequency. The input spectrum $S(\omega)$ must be continuous at the cutoff frequency ω_0 and indeed must be equal to zero at this point. Otherwise the hypothesis that $|G(\omega)|^2/\omega^{2n}$ is a maximum in the frequency region below ω_0 would be contradicted.

The solution is here carried through for the case of white noise, $N(\omega) = N_0$. As has been indicated, the optimum transfer function for this case is given by

$$|G(\omega)|^2 = \frac{N_0}{N_0 + S(\omega)} \quad (10)$$

The quantity $|G|^2/\omega^{2n}$ is a constant in the frequency region between zero and ω_0 . Since $S(\omega)$, by hypothesis, is zero outside this region, $|G(\omega)|^2$ is unity for frequencies greater than ω_0 . Continuity at ω_0 then dictates that the game theory transfer function be given by

$$|G(\omega)|^2 = |1 - F(\omega)|^2 = \begin{cases} (\omega/\omega_0)^{2n} , & \omega \leq \omega_0 , \\ 1 , & \omega \geq \omega_0 . \end{cases} \quad (10a)$$

The cutoff frequency is determined from the condition that

$$\int_{-\infty}^{\infty} g(\omega) d\omega = 1 \quad (10b)$$

Substitution of Eq. (10a) into Eq. (10) gives for the game theory signal spectrum, $S(\omega)$

$$S(\omega) = \begin{cases} N_0 \left[(\omega_0/\omega)^{2n} - 1 \right], & \omega \leq \omega_0, \\ 0 & \omega \geq \omega_0. \end{cases} \quad (10c)$$

The relationship between $S(\omega)$ and $g(\omega)$ substituted into Eq. (10b) yields the cutoff frequency, ω_0 . Thus

$$\omega_0^{2n+1} = \frac{2n+1}{2n} \pi \left(\frac{M^2}{N_0} \right). \quad (10d)$$

The quantity (M^2/N_0) is a dimensional parameter which determines the error.

It is to be noted that $G(\omega)$ is the well known so-called semi-infinite slope. The phase for such a transfer function is given by Bode, ² Chap. XV. A knowledge of the phase enables one easily to obtain $F(\omega)$, the actual transfer function, from the law of cosines.

For other noise spectra the solutions are obtained similarly, although not as conveniently. The results are generally not as simple as those for white noise.

When the optimum transfer function is used, all of the mean square time derivatives of the output become infinite. The argument follows. It is to be noted from Eq. (1) that $N(\omega)|F(\omega)|^2$ and $S(\omega)|G(\omega)|^2$ enter the expression for the mean square miss symmetrically. Thus an integral equation in terms of $F(\omega)$ similar to Eq. (3) can be derived by replacing $G(\omega)$ by $F(\omega)$ and interchanging $S(\omega)$ and $N(\omega)$. Examination of this equation indicates that for the optimum transfer function

$$\begin{aligned} |F(\omega)|^2 N(\omega) &\longrightarrow \frac{\text{constant}}{\omega^2} \\ \omega &\longrightarrow \infty \end{aligned} \quad (11)$$

In terms of its power spectrum the mean square value of the n^{th} time derivative of a stationary random function is known to be given by

$\frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^{2n} S(\omega) d\omega$. Thus the mean square value of the n^{th} time derivative of the filter output is

$$\left\langle \left| \frac{d^n x(t)}{dt^n} \right|^2 \right\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^{2n} |F(\omega)|^2 [S(\omega) + N(\omega)] d\omega \quad (12)$$

When the optimum transfer function is used it is seen that because of the noise contribution all the mean square derivatives are infinite.

The minimization of σ^2 subject to the restriction that its n^{th} mean square time derivative be finite can be accomplished by the method of Lagrange multipliers. It is then desired to minimize $\sigma^2 + \lambda \left\langle \left| \frac{d^n x(t)}{dt^n} \right|^2 \right\rangle$.

Here λ is a Lagrange multiplier, the value of which determines the value of the derivative. This process is especially simple if only the contribution due to noise (the portion which causes the divergence) is limited. It is necessary then to minimize

$$\begin{aligned} \sigma^2 + \lambda \left\langle \left| \frac{d^n x(t)}{dt^n} \right|^2 \right\rangle \\ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ S(\omega) |1 - F(\omega)|^2 + N(\omega) [1 + \lambda \omega^{2n}] |F(\omega)|^2 \right\} d\omega \quad (13) \end{aligned}$$

For the case of greatest interest, white noise, it is noted that the problem is formally equivalent to minimizing a fictitious σ^2 with a given input signal and a polynomial noise spectrum.

The game theory error has been computed for the case where the mean square second derivative of both the input signal and the output are limited. The game theory error, normalized, has been plotted in Fig. 4 as a function of the ratio of the root mean square second derivative of the output due to noise to the root mean square second derivative of the input signal.

MEAN SQUARE GAME THEORY ERROR
AS A FUNCTION OF THE RATIO OF THE
RMS OUTPUT 2nd DERIVATIVE DUE TO
NOISE (DENOTED AS M_o) TO THE RMS
SIGNAL 2nd DERIVATIVE

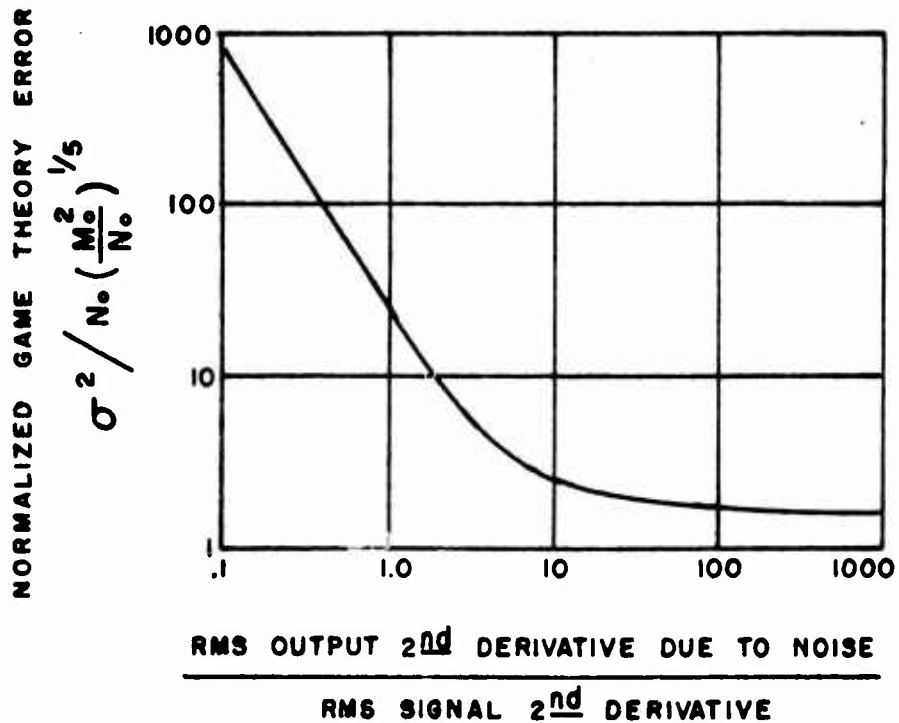


FIGURE 4

Acknowledgement

The authors wish to thank A. G. Carlton whose idea it was initially to treat this problem from the viewpoint of game theory. His assistance and suggestions have been invaluable.

References

1. N. Wiener, Extrapolation, Interpolation, and Smoothing of Stationary Time Series - (John Wiley and Sons, Inc., New York, 1950).
2. See e. g., H. W. Bode, Network Analysis and Feedback Amplifier Design - (D. Van Nostrand Company, Inc., New York, 1945), Chap. XIV.